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Incorporation and Test of Diffusion and Strain Effects in the Two-Dimensional Vortex Blob Technique

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Vorticity diffusion as well as small scale strain effects in a rotational fluid layer are incorporated in the vortex blob technique by changing the blob size. While the strain effect is computed by approximating the derivative of the normal velocity component across the layer, the vorticity diffusion is accounted for on the basis of the similarity solution for a diffusively spreading vortex sheet. This approach, which presents an alternative to the random walk method, allows us to track vorticity fronts in flows that do not conserve circulation such as miscible multiphase Hele–Shaw flows. We test the numerical method by comparing numerically obtained linear growth rates for an inviscid parallel shear layer and a miscible displacement process in a porous medium to exact results obtained from linear stability theory. When compared to the constant size vortex blob technique, the present extensions yield greatly improved results for the inviscid shear layer. For the diffusively spreading rotational layer arising in the miscible displacement process, the numerically obtained time-dependent growth rates show good quantitative agreement with exact results. © 1989 Academic Press, Inc.

1. INTRODUCTION

The evolution of the vorticity field in incompressible inviscid flow is governed by the theorems of Kelvin and Helmholtz [1]. This has stimulated the development of Lagrangian vortex methods for the purpose of numerically simulating such flows [2, 3]. Rosenhead [4] was the first one to discretize a vortex sheet into point vortices and to numerically compute its rollup. While the point vortex method represents an exact solution of the Euler equations, this highly singular form of discretizing the vorticity field leads to problems regarding the accuracy and stability of long-time integrations [5]. Furthermore, the vorticity in a flow is usually spread out over layers of finite thickness. As a result, more recent applications have replaced the point vortices by vortex blobs of finite cores, e.g., Chorin [6]. The convection velocity of the vortex blobs is usually approximated by the velocity induced at the blob center, although higher accuracy schemes exist, cf. Leonard [2]. Their usually circular shape is maintained throughout the simulation, and a

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smooth vorticity distribution is achieved by letting the cores overlap. In this way, the vortex blobs are not viewed as isolated patches of vorticity but rather as collectively forming a rotational layer of fluid. However, as pointed out by Leonard [2], the constant core shape and size of the blobs do not allow for the vorticity field to deform according to the local strain field of the flow. This effect becomes noticable, for example, when one tries to calculate the growth rates of the Kelvin–Helmholtz instability of a finite thickness shear layer discretized into one row of vortex blobs. Due to the finite thickness of the velocity profile, there is a short wavelength cutoff and a wavenumber of maximum amplification. While the growth rates obtained analytically from inviscid linear stability theory [7] are approximately reproduced for long wavelength perturbations, constant size vortex blob simulations [8] and stability calculations [9] show that the results for the growth rates of short waves are completely unsatisfactory. The reason for this lies in the fact that the evolution of the vorticity distribution is more complicated than that which can be represented by a mere displacement of the blobs. Under the influence of the induced strain, the vortical layer is subject to local thickening and thinning, which counteracts the growth of short waves. A discretization of the shear layer into many layers of vortex blobs [10], on the other hand, reproduces these effects and can be expected to result in better agreement with the analytical results, the price being higher computational costs. In this note, we describe and test an extension of the vortex blob method that captures the main effects of the local strain using one layer of vortex blobs. This is achieved by allowing the core radii of the blobs to vary as a result of the strain, in a way similar to the one briefly discussed in [11].

The goal of simulating high but finite Reynolds number flows based on their vorticity dynamics lead Chorin [6] to incorporate slight viscous effects into the vortex blob method. This was achieved by superimposing a random walk component on the purely inviscid motion calculated from the Biot–Savart law, thus simulating the diffusion of vorticity. While this approach has subsequently demonstrated its ability to reproduce many of the large scale features of turbulent flows (e.g., [12]), its applicability seems to be limited to simulations of flows in which the circulation is conserved. However, for many flows these conservation laws do not hold, and knowledge of the local flow features is needed in order to update the vorticity distribution, such as in vortex dynamics simulations of the Saffman–Taylor instability (e.g., [13]). As Tan and Homsy [14] show, it is very important for an accurate simulation of miscible displacement processes in Hele–Shaw flows to capture the dynamics that lead to the steepening of the concentration profile and the related vorticity front. As a result, we cannot deal with these flows by using constant size vortex blobs and instead we have to use a new approach to account for diffusion of vorticity. This diffusion of vorticity is the physical mechanism that causes very short wavelengths to be damped, so that this type of flow, just like the finite thickness shear layer, exhibits a wavenumber of maximum growth and a short wavelength cutoff. The method to be discussed and tested below represents the diffusion of the vortical layer through diffusive growth of the vortex blob cores, as suggested by Leonard [2]. Our interest thus focuses on flows in which physical mechanisms determine a wavelength of maximum growth as well as a short wavelength cutoff. Hence it is essential that a good numerical technique reproduces these wavelengths accurately. Such problems are fundamentally different from the rollup of an inviscid zero thickness shear layer, which has been the focus of a large body of literature of its own; see, e.g., Krasny [15]. Since in that case the flow does not exhibit a characteristic length scale, i.e., no physical mechanism to provide a wavelength of maximum amplification, Krasny's work is concerned with a purely numerical smoothing procedure and its convergence to the zero thickness problem. It should furthermore be emphasized that our interest focuses on the development of a front tracking method, while problems involving distributed vorticity might favor different approaches.

2. Numerical Algorithm

In the following, we want to compute the evolution of a flow that has a rotational layer of fluid discretized into a row of N vortex blobs of radius σ_i and circulation Γ_i, as shown in Fig. 1. The traditional approach for the numerical computation of the vorticity layer evolution calculates the velocity induced at the center of each vortex blob by all vortex blobs in the flow field and subsequently moves the blobs over a finite time step, leaving their shape and size unchanged. In the following, we describe and test the implementation of vorticity diffusion and of strain effects influencing the thickness of the layer.

(a) *The Effect of the Strain Field*

Our goal is to take into account the local thickening and thinning of the vorticity layer, which results from the strain induced during the instability of a rotational fluid layer. We keep in mind that we do not think of the vortex blobs as representing isolated patches of vorticity but rather as collectively forming a rotational fluid layer. Since we use overlapping blobs in order to obtain a smooth vorticity distribution, we do not have to conserve the size of the individual blob and can focus rather on capturing the strain effect on the layer thickness. This can be achieved by allowing the radius of each vortex blob to vary according to the local strain field. For this purpose, we need to evaluate the velocity component u_n perpendicular to the layer and its derivative in the direction x_n normal to the layer. This component $\partial u_n/\partial x_n$ of the strain tensor is responsible for the thickening and thinning of the rotational fluid layer. It is obtained in the following way: First we fit a cubic spline through the centers of all vortex blobs of the vorticity layer and thus obtain the direction of its centerline everywhere. As a next step, we find the points (x_{li}, y_{li}), (x_{ri}, y_{ri}) on the envelope of the vorticity layer as those being one core radius σ_i away from the center of blob i in the direction perpendicular to the layer's centerline (Fig. 1). We can now calculate the velocity components u_n at (x_{li}, y_{li}), (x_{ri}, y_{ri}) and approximate $\partial u_n/\partial x_n$ as $(u_n(x_{li}, y_{li}) - u_n(x_{ri}, y_{ri}))/2\sigma_i$. The resulting

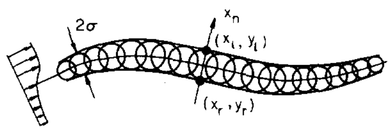

FIG. 1. A rotational fluid layer discretized into vortex blobs of radius σ_i. A cubic spline fit through all the blob centers gives the direction of the layer everywhere. The component $\partial u_n/\partial x_n$ of the strain tensor is responsible for the local thickening and thinning of the vortical layer and can be approximated using the velocities at the envelope of the layer and the blob radius.

change in the core radius of vortex blob i over the time step Δt is then approximately

$$\Delta\sigma_i = 0.5 \cdot (u_n(x_{li}, y_{li}) - u_n(x_{ri}, y_{ri})) \cdot \Delta t.$$

Alternatively, $\partial u_n/\partial x_n$ could be evaluated from the derivative of the tangential velocity component along the centerline and the continuity equation.

As mentioned above, the main goal of the present work is to develop a front tracking algorithm for flows that do not conserve circulation. However, because of the availability of stability results for comparison purposes, we will, as an example, compute the evolution of an inviscid parallel shear flow, which is governed by the Euler equations. If we assume the flow to be periodic in the x-direction and the velocity profile across the layer is of the form

$$u(y) = 0.5 \cdot (1 + \mathrm{erf}(y/\sigma))$$

the appropriate distribution function of the vorticity ω over the vortex blob as a function of the radial coordinate r is

$$\omega(r) = \Gamma_i/(\pi\sigma^2) \cdot \exp(-r^2/\sigma^2).$$

The circulation Γ_i of the vortex blob represents the integral over its vorticity distribution. We calculate the velocity induced at the center of each vortex blob and at the points on the envelope of the layer by all N vortex blobs in the control volume and their periodic images from

$$u(x, y) = \sum_{j=1}^{N} \left\{ \frac{y-y_j}{2\pi r^2} \Gamma_j \exp(-r^2/\sigma^2) - \frac{\Gamma_j \sinh(2\pi(y-y_j)/L)}{2L(\cosh(2\pi(y-y_j)/L) - \cos(2\pi(x-x_j)/L))} \right\}$$
$$v(x, y) = \sum_{j=1}^{N} \left\{ -\frac{x-x_j}{2\pi r^2} \Gamma_j \exp(-r^2/\sigma^2) + \frac{\Gamma_j \sin(2\pi(x-x_j)/L)}{2L(\cosh(2\pi(y-y_j)/L) - \cos(2\pi(x-x_j)/L))} \right\} \tag{1}$$

Here L is the x-dimension of the periodic control volume under consideration, and all but the closest image of the blobs are treated as point vortices. An initial sine wave perturbation of amplitude 10^{-8} displaces the vortex blob centers in the

y-direction. The time-step is taken as 0.1. Test calculations showed that a reduction of the time-step by a factor of ten changed the results by less than one percent. We monitor the displacement y_m of the vortex blob with the maximum initial perturbation. The growth rate can then be calculated from the values of y_m before and after every time-step. Since our initial perturbation does not duplicate the eigenfunction exactly, the computed growth rate converges to a steady value only after a few time-steps. Subsequently it remains constant to at least three digits for a time period during which the perturbation grows by several orders of magnitude.

Below we compare the growth rates obtained by taking strain effects into account in the way described above to those obtained with constant size cores as well as to the analytical ones given by Nakamura *et al.* [10] (Fig. 2). The curve labeled 1 represents the analytical growth rates as computed from the Rayleigh equation by Nakamura *et al.* [10]. Curve 2 shows the numerical results obtained by taking into account the local thickening and thinning of the layer. The spacing of the vortex blobs in the streamwise direction is approximately one half of a core radius. As a result, the calculation of the growth rate for the wavenumber $k = 0.1$ employed 109 vortex blobs, while the calculation for $k = 1.0$ used 9 vortex blobs. Corresponding calcultions for a spacing of one quarter and one eighth of a core radius resulted in a change of the growth rate of less than two percent. The maximum deviation occurred for short waves, while for long waves even a relatively large spacing of the blobs yields converged results. Curve 3 shows results obtained with constant size vortex blobs. Again the spacing is approximately one half of a core radius; a smaller spacing does not change the growth rate by more than two percent. It is obvious that the inclusion of the strain effects results in much better agreement with the

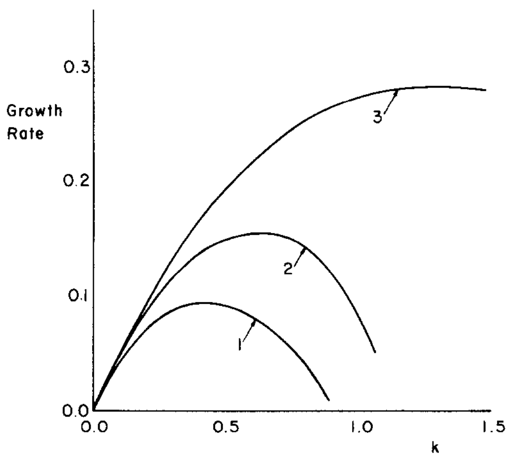

FIG. 2. Growth rates for the Kelvin–Helmholtz instability instability of an inviscid parallel shear layer with an error function velocity profile vs. wavenumber k. 1. analytical results [10]; 2. converged numerical results obtained with the vortex blob technique taking into account the local thickening and thinning of the vortical fluid layer; 3. converged numerical results obtained with constant size vortex blobs.

analytical results. Although the maximum growth rate is still about 50% too large, both the cutoff wavelength and the wavelength exhibiting maximum growth are approximately reproduced. The fixed size vortex blob calculation, on the other hand, does not show a clear maximum or a cutoff wavelength. The presence of a cutoff wavelength is an important feature, since even if a calculation initially only contains long waves, nonlinearity will give rise to short wavelength components. As a result, the presence of a cutoff wavelength has to be seen as a significant improvement over the fixed size blob method, even though the quantitative agreement of the variable blob size technique with linear stability theory is not completely satisfactory. We must keep in mind that the current numerical model is still relatively crude, as the shape of the vorticity distribution function over the vortex blob remains unchanged and, close to the cutoff, one wavelength is less than three times the average vortex blob diameter. Under these circumstances, the above degree of quantitative agreement seems to be the best we can hope for.

(b) *The Effect of Vorticity Diffusion*

In order to allow for the vorticity to spread diffusively, we apply a splitting algorithm that treats the convective and diffusive components of the vorticity evolution separately. This approach is similar to the one taken by Chorin [6]; however, his random walk technique was developed for the inclusion of viscous effects in flows that conserve circulation. Since our interest lies in the simulation of vorticity dominated flows that do not conserve circulation, we have to take diffusive effects into account in a different manner. The numerical approach will be explained for the example of a diffusively spreading parallel shear flow in which the vorticity satisfies a convective-diffusive equation with the transport coefficient ν. First we evaluate the velocities at the centers of the vortex blobs from the Biot–Savart law according to (1) and convect the vortex blobs over the time step Δt, also taking into account the strain effect. In a second step, we model the diffusive effects by performing a local 1-dimensional analysis of the vorticity layer at the location of each vortex blob. For this purpose, we treat the vorticity layer at the location of a vortex blob of radius σ_i as if it had the thickness $2\sigma_i$ everywhere and had evolved from the diffusive spreading of a plane vortex sheet, which is known to proceed according to the similarity solution

$$\omega(y, t) = 0.5/(\pi \nu t)^{1/2}\, e^{-(y^2/4\nu t)}.$$

Consequently, we can calculate a hypothetical age t_{1i} of the vorticity layer at the location of blob i as

$$t_{1i} = \sigma_i^2/4\nu.$$

As a result, the change in thickness of the vorticity layer over the time step Δt can then be calculated as

$$\Delta\sigma_i = \sigma_i((1 + \Delta t/t_{1i})^{1/2} - 1).$$

We finally obtain the thickness of the vorticity layer at the location of vortex blob i, i.e., the diameter of vortex blob i, at the end of the time step by superimposing the changes in thickness resulting from the straining and the diffusive effects.

As a test case for the time-splitting algorithm described above, we have calculated the linear growth rates of a rectilinear miscible displacement process in a porous medium in which a less viscous fluid penetrates a more viscous one. This flow, which is governed by Darcy's law, is known to be unstable, resulting in the formation of viscous fingers (for an overview, see Homsy [16]). A vortical fluid layer forms where two miscible components diffuse into each other, whereas the rest of the flow field remains irrotational. Hence, a front tracking algorithm based on the vorticity dynamics of the flow offers certain advantages. The circulation is no longer conserved as for inviscid flow, and instead it depends on the continuously changing velocity and concentration fields (details are given in [17]). Nonlinear calculations by Tan and Homsy [14] demonstrate the importance of strain and diffusion effects on the dynamics of the displacement process. They control the degree of steepness of the concentration profile and the related vorticity front, and their balance determines whether or not tip-splitting occurs. This in turn affects the sweep efficiency of the whole displacement process. Hence it is obvious that these flows cannot be dealt with on the basis of a front representation by a layer of vortex blobs of constant size. A rigorous treatment of the corresponding linear stability problem is given by Tan and Homsy [18], and we will compare our numerical results to their exact results. They derive and discuss the appropriate scaling of the problem as well as the definition of the growth rate, which is time-dependent due to the fact that the base flow changes continuously as a result of diffusion. Figure 3 shows the growth rates obtained from the quasi-steady-state approximation of Tan and Homsy [18] as well as those of the vortex blob simulation for periodic perturbations of three different wavelengths. As in the previous case, we start from a wavy initial displacement of the vortex blobs. We have checked the convergence of our numerical results by reducing the blob spacing as well as the time-step. The mobility ratio of the two fluids is 20.09 for all cases. The curve labeled 1 shows the exact results of Tan and Homsy [18]. Curve 2 gives the numerically obtained growth rates of the displacement amplitude, while curve 3 displays the numerically obtained values of the velocity growth rates. A calculation employing constant size blobs, on the other hand, would yield growth rates that do not vary in time and depend on the initial thickness of the rotational layer. We find very good quantitative agreement between the variable blob size calculations and the exact results for all but very short times, at which the quasi-steady-state approximation cannot be expected to hold. As for the inviscid shear layer analyzed above, the quantitative agreement improves for longer waves.

As the calculation continues in time, nonlinear effects become more important as the amplitude of the evolving fingers grows. Figure 4 compares the periodic array of fingers obtained from a variable size vortex blob calculation with results of the spectral code of Tan and Homsy [14]. While the evolving fingers have nearly identical shapes, the fingers in the vortex dynamic dynamics calculation develop slightly

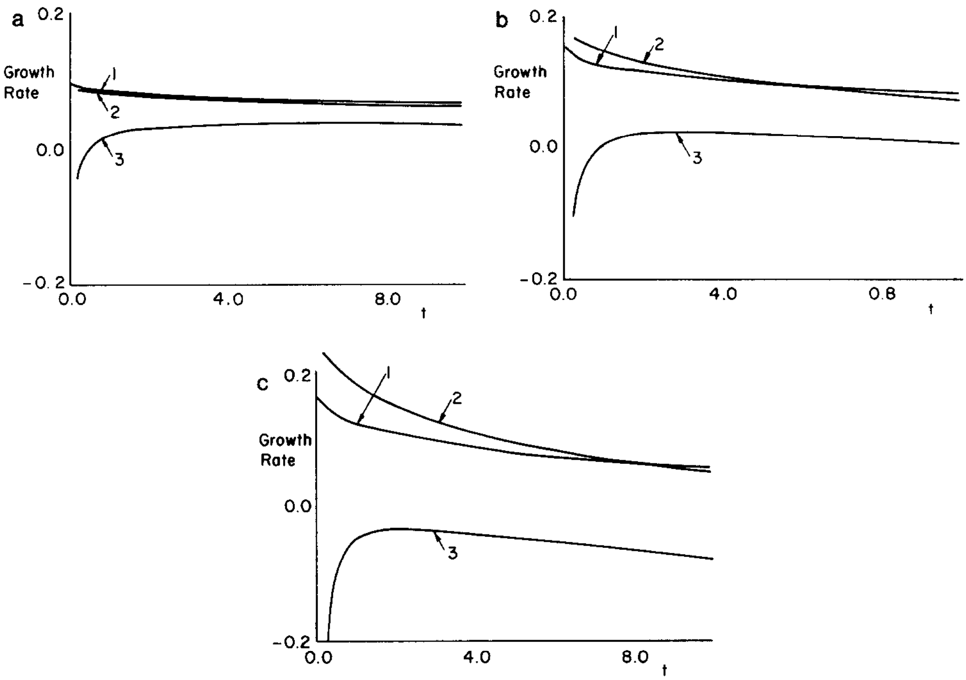

FIG. 3. Growth rates for the viscous fingering instability of a rectilinear miscible displacement process in a porous medium vs time. (a), (b), and (c) represent results for wavenumbers 0.1, 0.2, and 0.3, respectively: 1. exact results obtained for the linear stability problem via the quasi-steady-state approximation [18]; 2. numerically obtained growth rates of the displacement amplitude of the centerline of the vortical fluid layer; 3. numerically obtained growth rates of the velocity disturbance. The vortex dynamics computation yielding the growth rates 2 and 3 accounts for the effects of strain and diffusion. Calculations employing constant size vortex blobs would yield growth rates that do not vary in time and depend on the thickness of the rotational layer.

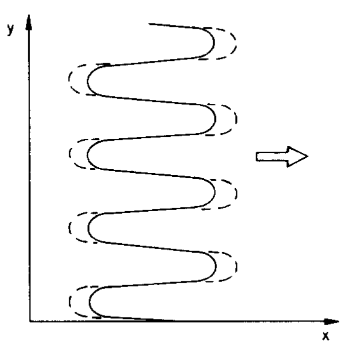

FIG. 4. Comparison of nonlinear finger shapes at identical times: ———, vortex dynamics calculation; – –, spectral method simulation (courtesy of C.-T. Tan and G. M. Homsy). While the finger shapes are nearly identical, the vortex dynamics calculation develops slightly slower in time.

slower in time. However, it is obvious that the variable size vortex blob technique is well suited for studying the dynamics of miscible viscous fingering, especially in geometries for which spectral collocation methods for front tracking problems are difficult to design.

In conclusion, the extensions of the vortex blob method discussed above allow us to capture the main effects of the strain field and the vorticity diffusion on the evolution of the rotational fluid layer even for flows that do not conserve circulation. In this way, the vortex blob front tracking technique gains the ability of simulating important dynamical features in an efficient manner. For the test case of an inviscid parallel shear layer, our technique reproduced the analytical growth rates much more closely than the constant size vortex blob method. Furthermore, it gives a realistic cutoff length. For a miscible displacement process governed by Darcy's law, our technique showed good quantitative agreement for the linear growth rates. We have validated our numerical method against know results from linear stability theory and reproduced nonlinear finger shapes observed in spectral calculations.

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